Technical Notes

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Coordinate Transformation for Laminar and Turbulent Boundary Layers

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Introduction

IT is well known that turbulent boundary layers are characterized by two transverse length scales—the boundary-layer thickness and the wall-layer thickness. These two length scales are generally quite different in magnitude and rate of growth, thereby making the analysis of turbulent layers more complicated than laminar boundary layers, where usually only one length scale is present, the boundary-layer thickness. For laminar flow, the Levy-Lees transformation[1] very effectively captures the boundary-layer growth, thereby significantly simplifying the analysis and solution of the governing equations. For turbulent flow, no equivalent transformation has yet been devised, leading to the evolution of ad hoc numerical schemes in which the grid point locations are adjusted to attempt to approximately capture the two different scale growths in the boundary layer. In the present investigation, a composite transformation, based entirely on fluid dynamic concepts, is given for the first time for the simultaneous capture of the two turbulent length scales.§ The utility of the approach is demonstrated through its application in a finite-difference solution of the boundary-layer equations in which the scheme is shown to be uniformly applicable to laminar, transitional, and turbulent flows.

Analysis

The approach taken here follows three principal steps: 1) a transformation is employed for the capture of the boundary-layer thickness using a turbulent generalization of the Levy-Lees variables; 2) a coordinate is established for the capture of a turbulent wall-layer region; and 3) a composite coordinate is established using matching concepts.

With regard to the boundary-layer growth, it was recognized by Prandtl and his students[3] that turbulent wake and jet flows are reducible to a laminar form through a simple coordinate transformation provided that the eddy viscosity is assumed constant. This concept was generalized by Clauser[4] to incompressible equilibrium turbulent boundary layers by establishing coordinates such that the outer layer portion (the wake-like region) could be reduced to a laminar similarity form. This concept has been extended by Werle and Verdon[5] to the general compressible case by replacing the laminar edge viscosity coefficient with an effective turbulent viscosity coefficient, which results in a turbulent flow version of the Levy-Lees variables. The modified Levy-Lees variables are obtained by transforming the surface coordinates s and n to new variables ξ and η defined by

$$\xi = \int_0^s \rho_e u_e \mu_{T_e} r_0^{2j} \mathrm{d}s \qquad \eta = \frac{\rho_e u_e r_0^j}{\sqrt{2\xi}} \int_0^n \frac{\rho}{\rho_e} \mathrm{d}n \tag{1}$$

where ρ_e and u_e are the boundary-layer edge values of density and streamwise velocity, r_0 is the surface transverse radius for axisymmetric flow, and $j=0$ and 1 for two-dimensional or axisymmetric flow, respectively. The definition of the "edge" turbulent viscosity term, μ_{T_e}, was established to capture the growth of the wake-like outer region of the boundary layer by writing

$$\mu_{T_e} \equiv \mu_e [1 + (\epsilon/\mu)]_{\mathrm{ref}} \tag{2}$$

where μ_e is the edge value of the molecular viscosity coefficient and $(\epsilon/\mu)_{\mathrm{ref}}$ is a representative outer region value of the eddy viscosity level (note $\epsilon_{\mathrm{ref}}=0$ for laminar flow). As discussed later, introduction of these coordinates into the compressible turbulent boundary-layer equations reduces them to a laminar similarity form in all but a very thin wall-layer region.

To accommodate this critical inner layer region, a new normal coordinate, N_i, is introduced based on the approximate analytical velocity profile expression presented by Whitfield[6]

$$N_i \approx \frac{u}{u_e}\bigg|_i = \frac{\sqrt{C_{f_e}/2}}{a} \tan^{-1}(a\sqrt{Re C_{f_e}} \xi \eta) \tag{3}$$

where C_{f_e} is the local skin-friction coefficient, Re the reference Reynolds number, and a an empirical constant equal to 0.09.

An additive composite coordinate for the inner and outer layer regions is now formed by first defining an outer region normal coordinate. The outer coordinate transformation is guided by the pioneering work of Clauser[4] in which he deduced universal coordinates such that correlated laminar similarity solutions for various wall slip velocities were shown to be in excellent agreement with the wake portion of equilibrium turbulent boundary layers measured experimentally. Hence, the outer transformation is deduced from an approximate fit to the Clauser correlation shown in Fig. 28 of Ref. 4, which is written as

$$N_o \approx \frac{u}{u_e}\bigg|_o = (1-b)\tanh^{1/\alpha}\left\{d\left[(1-b)\frac{\eta}{\eta^*}\right]^\alpha\right\} + b \tag{4}$$

This equation has a "slip velocity" b at $\eta=0$ to allow for the presence of the inner region. The quantities α and d were set equal to 1.25 and 0.45, respectively. Whitfield found that this functional form closely fits a wide range of turbulent data in the outer region, which gives further support to the idea that laminar and turbulent solutions resemble each other in this region if they are expressed in the appropriate variables. Equation (4) also contains the transformed incompressible displacement thickness

$$\eta^* = \int_0^\infty \left(1 - \frac{u}{u_e}\right) \mathrm{d}\eta \tag{5}$$

The coordinate transformation is completed by forming an

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additive composite coordinate given by

$$N = N_i + N_o - N_{i/o} \tag{6}$$

where $N_{i/o}$ denotes the outer limit of the inner coordinate which is matched to the inner limit of the outer expansion thereby yielding $N_{i/o} = b = \pi/2a\sqrt{C_{f_e}/2}$.

In these new coordinates ξ, N, the governing equations now become

$$2\xi(F_\xi + N_\xi F_N) + F + N_\eta V_N = 0 \tag{7}$$

$$2\xi F F_\xi + (VN_\eta + 2\xi F N_\xi)F_N = \beta(g - F^2) + N_\eta[\bar{\mu} N_\eta F_N]_N \tag{8}$$

$$2\xi F g_\xi + (VN_\eta + 2\xi F N_\xi) g_N = \alpha N_\eta [\hat{\mu} N_\eta (F^2/2)_N]_N + N_\eta[\bar{k} N_\eta g_N]_N \tag{9}$$

in which the following definitions have been used:

$$F = \frac{u}{u_e} \quad V = \frac{2\xi}{\rho_e \mu_{T_e} r_0^{2j}}\left(\frac{\rho v}{\sqrt{2\xi}} + F\eta_s\right) \quad g = \frac{H}{H_e}$$

$$\beta = \frac{2\xi}{M_e}\frac{dM_e}{d\xi} \quad \alpha = \frac{(\gamma - 1)M_e^2}{1 + \dfrac{\gamma - 1}{2}M_e^2} \quad \mu_T = \mu\left(1 + \frac{\epsilon}{\mu}\right)$$

$$\bar{\mu} = \frac{\rho\mu_T}{\rho_e \mu_{T_e}} \quad \hat{\mu} = \left(1 - \frac{1}{Pr}\right)\frac{\rho\mu}{\rho_e\mu_{T_e}} \quad \bar{k} = \frac{1}{Pr}\frac{\rho\mu}{\rho_e\mu_{T_e}}\left(1 + \frac{\epsilon}{\mu}\frac{Pr}{Pr_t}\right) \tag{10}$$

A noniterative implicit finite-difference scheme has been applied to these equations in which a uniform mesh is used in the normal coordinate N direction. The use of the coordinate transformation results in less than a 10% increase in computer time over that using standard Levy-Lees variables ($\epsilon_{ref} = 0$) as employed in the UTRC computer code recently developed[7] using the variable grid finite-difference scheme developed by Blottner.[8] This code has been used in the present work to provide calculations for comparison purposes. Note that the new coordinate transformation is an adaptive grid procedure since it relies on the local values of the reference viscosity, skin-friction coefficient, and the displacement thickness to complete the specification of the composite coordinate at each streamwise location. In the results presented here, these quantities were obtained from the numerical solution at the previous streamwise station since a noniterative scheme was used. This adaptive grid procedure is applicable to laminar flow since the wall-layer region is nonexistent (hence $N_i = 0$) and only the outer transformation is used. In transitional flows the wall-layer region is initiated at the start of transition, thus allowing for the natural development of the wall region as the flow evolves from a laminar to a fully turbulent boundary layer.

Results and Discussion

In order to test the present adaptive grid procedure for both positive and negative pressure gradients, the boundary-layer edge velocity distribution deduced experimentally by Schubauer and Klebanoff[9] for the flow over an "airfoil-like" body was prescribed for the present calculations. This u_e distribution is shown as the insert in Fig. 1 along with the boundary-layer velocity profiles that were deduced at various streamwise positions assuming the flow remained laminar. It is seen in this figure that the use of this new coordinate results in velocity profiles that are nearly correlated as they show only mild departures from the unit slope line $F = N$, even up to the laminar separation point.

Figure 2 shows the computed velocity profiles at the same streamwise locations as in Fig. 1 which result when transition from laminar to turbulent flow was assumed to occur between stations 1 and 2. Also shown in the insert in Fig. 2 is a

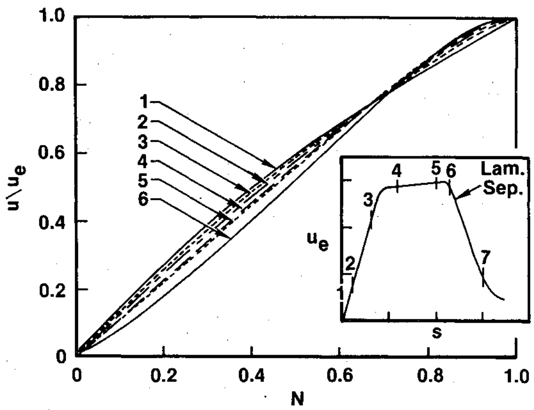

Fig. 1 Laminar velocity profiles in composite coordinate.

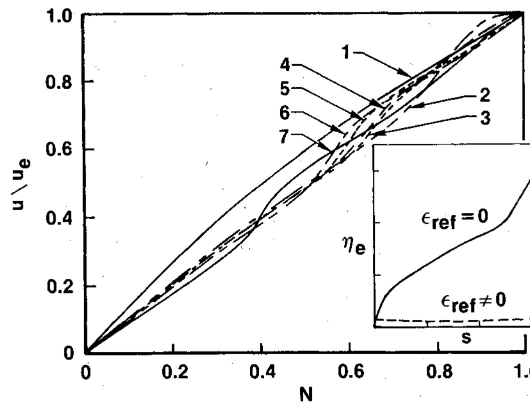

Fig. 2 Turbulent velocity profiles in composite coordinate.

comparison between the streamwise variation of the boundary-layer edge as deduced in the laminar Levy-Lees variable ($\epsilon_{ref} = 0$) with that obtained in terms of the turbulent Levy-Lees variable ($\epsilon_{ref} \neq 0$). The ability of the turbulent Levy-Lees variables to capture the turbulent boundary-layer growth is clearly seen here for a flow in which both strong favorable and adverse pressure gradients occur. Nonetheless, despite the capture of the turbulent boundary-layer growth, the boundary-layer profile expressed solely in terms of this turbulent Levy-Lees variable still requires a variable grid to adequately resolve the high-gradient wall-layer region. An improved treatment of the wall-layer region is made by introducing the composite coordinate transformation, Eq. (6), which enlarges the inner region, thereby permitting a uniform mesh to be used.

The computed profiles shown in Fig. 2 in terms of the composite coordinate N show the same 0(1) variation across the boundary layer as the laminar profile (station 1) which indicates that the relative streamwise growths of the wall and wake layers have been appropriately accounted for in this transformation. The inflection point in the turbulent profiles in Fig. 2 is the approximate location where the wall layer merges into the outer wake portion of the boundary layer. It is seen that this point moves toward the wall with increasing distance downstream due to the decrease in the skin-friction coefficient, which, as shown in Eq. (3), scales the thickness of the wall-layer region. It is shown in Ref. 2 that the skin-friction distribution predicted in the present calculation is in excellent agreement with the experimental result except in the aft strong adverse pressure gradient region where other investigators[10] have concluded that there are three-dimensional effects that are outside the scope of the present analysis. The results shown in Fig. 2 were obtained using 101 grid points across the boundary layer; however, the skin friction showed only a slight change (less than 5%) when the number of points was reduced to 20, thereby indicating the insensitivity to the mesh density since, as shown in Fig. 2, the velocity variation in the new composite coordinate is considerably reduced over that usually found in turbulent boundary layers.

Conclusion

In conclusion, a new composite coordinate transformation, based on well-known fluid dynamic concepts, has been presented which captures the boundary-layer thickness and simultaneously enlarges the wall-layer region. With this transformation, an adaptive grid procedure for numerical solution of the governing equations was demonstrated for laminar, transitional, and turbulent flow. The adaptive grid scheme presented here is simpler to use than a variable grid scheme since now only the total number of desired points needs to be specified by the user.

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Free Energy of Random Sound Oscillations in Gases

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Introduction

In solids and liquids, the effect of sound waves on the thermodynamic quantities was studied by Landau and others.[1] In ionized gases, the electrical oscillations (plasma oscillations) affect the thermal equilibrium of the system.[2,3] Similarly, we are considering the contribution of the sound-wave oscillations to the free energy of noncondensed gases. In thermal equilibrium of gases, the acoustic oscillations share the partition of energy and thus change the thermodynamic functions of the system. The distribution of the sound-wave quanta is determined by Bose statistics, which are used herein.

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The problem under consideration is concerned with gases as a macroscopic continuum which exhibits a set of separate elementary exitations, that is, the sound-wave oscillations. These exitations behave as "quasiparticles" moving in the volume occupied by the gas and have definite energies. The free energy of the gas evaluated by the theory to be presented will take into consideration the energy of the random sound-wave oscillations, in addition to the random thermal energies of the gas particles. It will be shown that the effect of the sound waves is important only at high temperatures and high gas densities. The results of this theory are applicable at temperatures and densities for which the gas is not in a condensed state (liquid or solid).

Although nonideal effects due to finite particle size are not taken into account explicitly, it should be noted that the gas under consideration is not an ideal one. The existence of sound waves in the gas implies that there are particle interactions, since a gas cannot perform the ordered, collective mean mass motions of random sound waves without such interactions.

Theory

Consider a gas as a continuum of volume V containing N atoms. Because the velocity of the gas in a sound wave is in the direction of propagation, the sound waves are longitudinal. Each oscillator of frequency ω_σ of the longitudinal sound waves can have only the energies

$$E_{n_\sigma}=\hbar\omega_\sigma(n_\sigma+\tfrac{1}{2}),\quad n_\sigma=0,1,2,\ldots\infty \tag{1}$$

where $\hbar=h/2\pi$ = reduced Planck constant.

The frequency of the sound waves with wave number k_σ is

$$\omega_\sigma=k_\sigma C_s,\quad C_s=(\gamma KT/M)^{1/2} \tag{2}$$

where $\gamma\cong C_p/C_v$ is the polytropic coefficient and M = mass of atoms. Accordingly, the partition function of the gas oscillations is

$$Z=\prod_\sigma\sum_{n_\sigma=0}^{\infty}e^{-\beta\hbar C_s k_\sigma(n_\sigma+1/2)}=\prod_\sigma\frac{e^{-\beta\hbar C_s k_\sigma/2}}{1-e^{-\beta\hbar C_s k_\sigma}} \tag{3}$$

in which $\beta=1/KT$, where K is the Boltzmann constant and T the temperature of the system. From the partition function Z, the thermodynamic quantities such as pressure, internal energy, etc. are derived in the usual way. The free energy of the random sound oscillations is given by

$$\Delta F=-KT\ell n Z \tag{4}$$

In the limit $V\to\infty$, the discrete eigenfrequencies ω_σ are replaced by a continuous spectrum $\omega=\omega(k)$ in accordance with the dispersion law for sound waves of wave length $\lambda=2\pi/k$,

$$\omega=kC_s,\qquad 0\le k\le\hat{k} \tag{5}$$

The theory to be presented is sensitive toward the cutoff wave number $\hat{k}$, which is large in all cases of interest. Since acoustic waves with wavelengths $\lambda<\max(\bar{r},L)$ and mean free paths $L<\bar{r}=n^{-1/3}$ are not possible in gases, $\hat{k}=2\pi/\hat{\lambda}$ is determined by the mean free path L,

$$\hat{k}=2\pi/L,\qquad L=L(n,T) \tag{6}$$

where $n=N/V$ is the density of the atoms. The number of (longitudinal) wave modes with wave numbers between k and $k+dk$ in volume V is $g(k)dk=V4\pi k^2dk/(2\pi)^3$. Accordingly, Eqs. (3) and (4) give

$$\Delta F=-\frac{KTV}{2\pi^2}\int_0^{\hat{k}}k^2\ell n\left(\frac{e^{-\beta\hbar C_s k/2}}{1-e^{-\beta\hbar C_s k}}\right)dk \tag{7}$$